Exercise 15

The transverse vibration of an infinite elastic beam of mass m per unit length and bending stiffness EI is governed by

$$u_{tt} + a^2 u_{xxxx} = 0, \quad a^2 = \frac{EI}{m}, \; -\infty < x < \infty, \; t > 0.$$

Solve this equation subject to the boundary and initial data

$$\begin{gathered} u(0,t) = 0 \quad \text{for all } t > 0, \\ u(x,0) = \phi(x), \quad \text{and} \quad u_t(x,0) = \psi''(x) \quad \text{for } 0 < x < \infty. \end{gathered}$$

Show that the Fourier transform solution is

$$U(k,t) = \Phi(k)\cos(atk^2) - \left(\frac{1}{a}\right)\Psi(k)\sin(atk^2).$$

Find the integral solution for $u(x,t)$.

Solution

The PDE is defined for $-\infty < x < \infty$, so we can apply the Fourier transform to solve it. We define the Fourier transform here as

$$\mathcal{F}\{u(x,t)\} = U(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ikx} u(x,t)\,dx,$$

which means the partial derivatives of u with respect to x and t transform as follows.

$$\begin{aligned} \mathcal{F}\left\{\frac{\partial^n u}{\partial x^n}\right\} &= (ik)^n U(k,t) \\ \mathcal{F}\left\{\frac{\partial^n u}{\partial t^n}\right\} &= \frac{d^n U}{dt^n} \end{aligned}$$

Take the Fourier transform of both sides of the PDE.

$$\mathcal{F}\{u_{tt} + a^2 u_{xxxx}\} = \mathcal{F}\{0\}$$

The Fourier transform is a linear operator.

$$\mathcal{F}\{u_{tt}\} + a^2\mathcal{F}\{u_{xxxx}\} = 0$$

Transform the derivatives with the relations above.

$$\frac{d^2U}{dt^2} + a^2(ik)^4 U = 0$$

Expand the coefficient of U.

$$\frac{d^2U}{dt^2} + a^2k^4 U = 0 \tag{1}$$

The PDE has thus been reduced to an ODE. Before we solve it, we have to transform the initial conditions as well. They are only defined for $0 < x < \infty$, so we can't transform them as they are.

In order to transform them and satisfy the homogeneous boundary condition $u(0,t)=0$, we have to consider the odd extensions of $\phi(x)$ and $\psi(x)$ to the whole line. That is,

$$\phi_{\text{odd}}(x)=\begin{cases}\phi(x) & 0<x<\infty\\ -\phi(-x) & -\infty<x<0\end{cases}$$
$$\psi_{\text{odd}}(x)=\begin{cases}\psi(x) & 0<x<\infty\\ -\psi(-x) & -\infty<x<0\end{cases}$$

Taking the Fourier transform of the extended initial conditions gives

$$\begin{aligned} u(x,0)=\phi_{\text{odd}}(x) \quad &\rightarrow \quad \mathcal{F}\{u(x,0)\}=\mathcal{F}\{\phi_{\text{odd}}(x)\}\\ &\qquad\quad U(k,0)=\Phi(k) \qquad (2)\\ \frac{\partial u}{\partial t}(x,0)=\psi''_{\text{odd}}(x) \quad &\rightarrow \quad \mathcal{F}\left\{\frac{\partial u}{\partial t}(x,0)\right\}=\mathcal{F}\{\psi''_{\text{odd}}(x)\}\\ &\qquad\quad \frac{dU}{dt}(k,0)=(ik)^2\Psi(k)=-k^2\Psi(k). \qquad (3)\end{aligned}$$

Equation (1) is an ODE in t, so k is treated as a constant. The solution to the ODE is given in terms of sine and cosine.

$$U(k,t)=A(k)\cos ak^2t+B(k)\sin ak^2t$$

Apply the first initial condition, equation (2).

$$U(k,0)=A(k)=\Phi(k)$$

In order to apply the second initial condition, differentiate $U(k,t)$ with respect to t.

$$\frac{dU}{dt}(k,t)=-ak^2A(k)\sin ak^2t+ak^2B(k)\cos ak^2t$$

Now apply equation (3).

$$\frac{dU}{dt}(k,0)=ak^2B(k)=-k^2\Psi(k) \quad\rightarrow\quad B(k)=-\left(\frac{1}{a}\right)\Psi(k)$$

Therefore, the solution to the ODE that satisfies the initial conditions is

$$U(k,t)=\Phi(k)\cos ak^2t-\left(\frac{1}{a}\right)\Psi(k)\sin ak^2t.$$

In order to change back to $u(x,t)$, we have to take the inverse Fourier transform of $U(k,t)$. It is defined as

$$\mathcal{F}^{-1}\{U(k,t)\}=u(x,t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}U(k,t)e^{ikx}\,dk,$$

which means it is a linear operator.

$$\begin{aligned}u(x,t)&=\mathcal{F}^{-1}\{U(k,t)\}\\ &=\mathcal{F}^{-1}\left\{\Phi(k)\cos ak^2t-\frac{1}{a}\Psi(k)\sin ak^2t\right\}\\ &=\mathcal{F}^{-1}\{\Phi(k)\cos ak^2t\}-\frac{1}{a}\mathcal{F}^{-1}\{\Psi(k)\sin ak^2t\}\end{aligned}$$

Because we are taking the inverse Fourier transform of products of two functions, we can apply the convolution theorem, which states that

$$\mathcal{F}^{-1}\{F(k)G(k)\} = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(\xi)g(x-\xi)\,d\xi.$$

Looking up the inverse Fourier transforms of $\cos ak^2t$ and $\sin ak^2t$ in a table,

$$\mathcal{F}^{-1}\left\{\cos ak^2t\right\} = \frac{1}{\sqrt{4at}}\left(\cos\frac{x^2}{4at} + \sin\frac{x^2}{4at}\right)$$
$$\mathcal{F}^{-1}\left\{\sin ak^2t\right\} = \frac{1}{\sqrt{4at}}\left(\cos\frac{x^2}{4at} - \sin\frac{x^2}{4at}\right),$$

we can write $u(x,t)$ by the convolution theorem as

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\phi_{\text{odd}}(\xi)\frac{1}{\sqrt{4at}}\left[\cos\frac{(x-\xi)^2}{4at} + \sin\frac{(x-\xi)^2}{4at}\right]d\xi - \frac{1}{a}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\psi_{\text{odd}}(\xi)\frac{1}{\sqrt{4at}}\left[\cos\frac{(x-\xi)^2}{4at} - \sin\frac{(x-\xi)^2}{4at}\right]d\xi.$$

Pull the constants in front of the integrals.

$$u(x,t) = \frac{1}{\sqrt{8\pi at}}\int_{-\infty}^{\infty}\phi_{\text{odd}}(\xi)\left[\cos\frac{(x-\xi)^2}{4at} + \sin\frac{(x-\xi)^2}{4at}\right]d\xi - \frac{1}{a}\frac{1}{\sqrt{8\pi at}}\int_{-\infty}^{\infty}\psi_{\text{odd}}(\xi)\left[\cos\frac{(x-\xi)^2}{4at} - \sin\frac{(x-\xi)^2}{4at}\right]d\xi.$$

We want the answer to be in terms of the given functions, ϕ and ψ, so substitute the expressions for ϕ_{odd} and ψ_{odd} by splitting up each integral into two: one going from $-\infty$ to 0 and one going from 0 to ∞.

$$\begin{aligned}u(x,t) = \frac{1}{\sqrt{8\pi at}}&\left\{\int_{-\infty}^{0}[-\phi(-\xi)]\left[\cos\frac{(x-\xi)^2}{4at} + \sin\frac{(x-\xi)^2}{4at}\right]d\xi + \int_{0}^{\infty}\phi(\xi)\left[\cos\frac{(x-\xi)^2}{4at} + \sin\frac{(x-\xi)^2}{4at}\right]d\xi\right\}\\ &- \frac{1}{a}\frac{1}{\sqrt{8\pi at}}\left\{\int_{-\infty}^{0}[-\psi(-\xi)]\left[\cos\frac{(x-\xi)^2}{4at} - \sin\frac{(x-\xi)^2}{4at}\right]d\xi + \int_{0}^{\infty}\psi(\xi)\left[\cos\frac{(x-\xi)^2}{4at} - \sin\frac{(x-\xi)^2}{4at}\right]d\xi\right\}\end{aligned}$$

Make the substitution, $s=-\xi$ and $ds=-d\xi$, in the first and third integrals and make the substitution, $s=\xi$ and $ds=d\xi$, in the second and fourth integrals.

$$\begin{aligned}u(x,t) = \frac{1}{\sqrt{8\pi at}}&\left\{\int_{\infty}^{0}\phi(s)\left[\cos\frac{(x+s)^2}{4at} + \sin\frac{(x+s)^2}{4at}\right]ds + \int_{0}^{\infty}\phi(s)\left[\cos\frac{(x-s)^2}{4at} + \sin\frac{(x-s)^2}{4at}\right]ds\right\}\\ &- \frac{1}{a}\frac{1}{\sqrt{8\pi at}}\left\{\int_{\infty}^{0}\psi(s)\left[\cos\frac{(x+s)^2}{4at} - \sin\frac{(x+s)^2}{4at}\right]ds + \int_{0}^{\infty}\psi(s)\left[\cos\frac{(x-s)^2}{4at} - \sin\frac{(x-s)^2}{4at}\right]ds\right\}\end{aligned}$$

Place a minus sign in front of the first and third integrals to switch the limits of integration.

$$\begin{aligned}
u(x,t) = \frac{1}{\sqrt{8\pi a t}} \left\{ -\int_0^\infty \phi(s) \left[\cos\frac{(x+s)^2}{4at} + \sin\frac{(x+s)^2}{4at} \right] ds \right. \\
\left. + \int_0^\infty \phi(s) \left[\cos\frac{(x-s)^2}{4at} + \sin\frac{(x-s)^2}{4at} \right] ds \right\} \\
- \frac{1}{a}\frac{1}{\sqrt{8\pi a t}} \left\{ -\int_0^\infty \psi(s) \left[\cos\frac{(x+s)^2}{4at} - \sin\frac{(x+s)^2}{4at} \right] ds \right. \\
\left. + \int_0^\infty \psi(s) \left[\cos\frac{(x-s)^2}{4at} - \sin\frac{(x-s)^2}{4at} \right] ds \right\}
\end{aligned}$$

Combining the integrals and distributing the minus sign, we therefore have for the integral solution

$$\begin{aligned}
u(x,t) = \frac{1}{\sqrt{8\pi a t}} \int_0^\infty \left[\cos\frac{(x-s)^2}{4at} + \sin\frac{(x-s)^2}{4at} - \cos\frac{(x+s)^2}{4at} - \sin\frac{(x+s)^2}{4at} \right] \phi(s)\, ds \\
+ \frac{1}{a}\frac{1}{\sqrt{8\pi a t}} \int_0^\infty \left[\cos\frac{(x+s)^2}{4at} - \sin\frac{(x+s)^2}{4at} - \cos\frac{(x-s)^2}{4at} + \sin\frac{(x-s)^2}{4at} \right] \psi(s)\, ds.
\end{aligned}$$

If we plug in $x = 0$, then all the terms in the square brackets cancel out, so the homogeneous boundary condition is satisfied. This integral solution is valid for $x \geq 0$, as the initial conditions are only defined there.